SHENTON
COLLEGE

Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 & 4

Section One: Calculator-free

Student Name: Solution

Teacher: Ai Friday White

Time allowed for this section

Reading time before commencing work: five (5) minutes
Working time: fifty (50) minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisors **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	36%
Section Two: Calculator-assumed	12	12	100	93	64%
				Total	100

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4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. Do **not** place the Formula Sheet inside your Question/Answer booklet. It will be collected separately.

Markers Use Only		
Question	**Max.**	**Mark**
1	7	
2	6	
3	5	
4	7	
5	7	
6	7	
7	7	
8	6	
Section One Total	52	
Section One %		
Section Two Total	92	
Section Two %		
Overall Deductions	Units	
	Rounding	
	Notation	
Total	144	
Overall %		

Section One: Calculator-free **(52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

The function f is defined by $f(x) = \dfrac{x^2 - 5}{3 - x}$, $x \neq 3$.

The second derivative of f is $f''(x) = 8(3 - x)^{-3}$.

Determine the coordinates and nature of all stationary points of the graph of $y = f(x)$.

$$f'(x) = \frac{(3-x)(2x) - (x^2-5)(-1)}{(3-x)^2}$$

✓ Demonstrates correct use of Quotient Rule

stationary point when $f'(x) = 0$ ✓ correct $f'(x)$

i.e. $6x - 2x^2 + x^2 - 5 = 0$ ✓ $f'(x) = 0$

i.e. $-x^2 + 6x - 5 = 0$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 1$ or $x = 5$ ✓ determines x-coords of stat. points.

$f''(1) = \dfrac{8}{2^3} > 0 \therefore$ Minimum

$f''(5) = \dfrac{8}{(-2)^3} = \dfrac{8}{-8} < 0 \therefore$ maximum

✓ demonstrates correct use of $f''(x)$ to determine NATURE.

Minimum stationary point at $(1, -2)$ ✓ correct minimum

Maximum stationary point at $(5, -10)$ ✓ correct maximum

Question 2 **(6 marks)**

The continuous random variable X takes values in the interval 3 to 8 and has cumulative distribution function $F(x)$ where

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 3 \\ \dfrac{x-3}{5} & 3 \le x \le 8 \\ 1 & x > 8. \end{cases}$$

(a) Determine

(i) $P(X \le 4.5)$. (1 mark)

$$F(4{\cdot}5) = \frac{4{\cdot}5 - 3}{5} = \frac{1{\cdot}5}{5} = 0{\cdot}3$$

✓ correct probability

(ii) the value of k, if $P(X > k) = 0.75$. (2 marks)

$$P(X \le k) = 0{\cdot}25$$

$$\frac{k-3}{5} = 0{\cdot}25$$

$$k = 4{\cdot}25$$

✓ indicates use of $P(X \le k)$

✓ correct value of k.

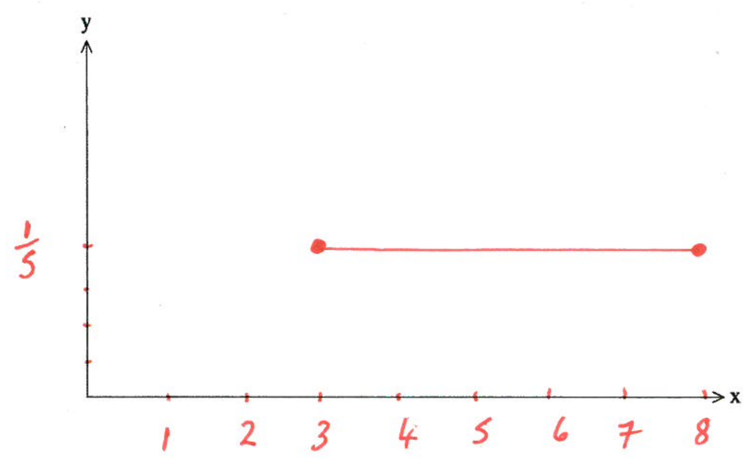
(b) Determine $f(x)$, the probability density function of X, and use the axes below to sketch the graph of $y = f(x)$. (3 marks)

$$\frac{d}{dx}\left(\frac{1}{5}x - \frac{3}{5}\right) = \frac{1}{5}$$

$$f(x) = F'(x) = \begin{cases} \frac{1}{5} & 3 \le x \le 8 \\ 0 & \text{otherwise.} \end{cases}$$

✓ $f(x)$

✓ draws $f(x) = \frac{1}{5}$ between endpoints



✓ scales axes

Question 3 **(5 marks)**

The rate of change of pressure in an air tank is given by $P'(t) = -2e^{-\frac{t}{25}}$, where t is the time in minutes since it began emptying from an initial pressure of 65 psi. *(psi is a unit of pressure expressed in pounds of force per square inch of area).*

(a) Determine an expression for the pressure P in the tank at any time t, $t \geq 0$. (2 marks)

$$P(t) = \int -2e^{-\frac{t}{25}}\,dt$$

$$= \frac{-2e^{-\frac{t}{25}}}{-\frac{1}{25}} + c$$

✓ correctly integrates $P'(t)$

$$= 50e^{-\frac{t}{25}} + c$$

$$P(0) = 50e^{0} + c = 65$$

$$c = 15$$

$$P(t) = 50e^{-\frac{t}{25}} + 15$$

✓ correct expression for $P(t)$

(b) Determine

(i) the exact time taken for the pressure in the tank to fall to 25 psi. (2 marks)

$$50e^{-\frac{t}{25}} + 15 = 25$$

$$50e^{-\frac{t}{25}} = 10$$

$$e^{-\frac{t}{25}} = 0.2$$

✓ simplifies equation to $e^{-\frac{t}{25}} = 0.2$

$$-\frac{t}{25} = \ln 0.2$$

$$-t = 25\ln 0.2$$

✓ correct time

units

$$t = -25\ln 0.2 \text{ minutes}$$

(ii) the long term behaviour of the pressure in the tank as $t \to \infty$ for $t \geq 0$. (1 mark)

As $t \to \infty$

$P(t) \to 15$ psi

✓ correct pressure

Question 4 **(7 marks)**

(a) Determine an expression for $f'(x)$ for each of the following functions.
DO NOT SIMPLIFY YOUR ANSWERS.

(i) $f(x) = \ln(1 - \cos 3x)$. (2 marks)

$$f'(x) = \frac{1}{1-\cos 3x} \cdot 3\sin 3x$$

$$= \frac{3\sin 3x}{1-\cos 3x}$$

✓ numerator or $\frac{d}{dx}(1-\cos 3x)$

✓ denominator.

(ii) $f(x) = e^{5x}(5 - 2x)^3$. (3 marks)

$$f'(x) = e^{5x} \cdot 3(5-2x)^2(-2) + (5-2x)^3\, 5e^{5x}$$

✓ $\frac{d}{dx} e^{5x} = 5e^{5x}$

✓ $\frac{d}{dx}(5-2x)^3$

✓ demonstrates use of product rule

(b) For the positive number x, let $A(x) = \int_0^x (8 - 2^{t^2})\, dt$.

Determine the value(s) of x for which $\frac{dA}{dx} = 0$. (2 marks)

$$\frac{dA}{dx} = \frac{d}{dx}\int_0^x (8-2^{t^2})\,dt$$

$$= 8 - 2^{x^2}$$

✓ expression for $\frac{dA}{dx}$ $A'(x)$

$$\therefore\ 8 - 2^{x^2} = 0$$

$$8 = 2^{x^2}$$

$$2^3 = 2^{x^2}$$

$$x^2 = 3$$

$$x = \sqrt{3}$$

✓ correct value of x.

Question 5 **(7 marks)**

(a) Simplify $3\log 5 + \log 4 - \log\frac{1}{2}$. (2 marks)

$\log 5^3 + \log 4 - \log\frac{1}{2}$

$= \log \frac{125 \cdot 4}{\frac{1}{2}}$

$= \log 1000$

$= 3$

✓ expresses as single log

✓ simplifies to number.

(b) Given that $\log_a x = 0.8$, determine the value of $\log_a\left(x^2\sqrt{x}\right)$. (2 marks)

$\log_a\left(x^2 x^{\frac{1}{2}}\right) = \log_a x^{2.5}$

$= 2.5 \log_a x$ ✓

$= 2.5(0.8)$

$= 2$

$2\log_a x + \frac{1}{2}\log_a x$ ✓

$= 2(0.8) + \frac{1}{2}(0.8)$

$= 2$

✓ demonstrates correct use of log laws

✓ correct value

(c) Determine the solution to the equation $4^{2x} = 3^{2-x}$ in the form $x = \dfrac{\log a}{\log b}$. (3 marks)

$4^{2x} = 3^{2-x}$

$2x\log 4 = (2-x)\log 3$ ✓

$2x\log 4 = 2\log 3 - x\log 3$

$2x\log 4 + x\log 3 = 2\log 3$

$x(2\log 4 + \log 3) = 2\log 3$ ✓

$x = \dfrac{\log 3^2}{\log 4^2 \cdot 3}$

$x = \dfrac{\log 9}{\log 48}$ ✓

✓ writes as log equation

✓ factors out x

✓ solves and simplifies into required form.

Question 6 **(7 marks)**

The discrete random variable X is defined by

$$P(X = x) = \begin{cases} \dfrac{2x + k}{3} & x = 0, 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Determine the value of the constant k. (2 marks)

$$\frac{2(0) + k}{3} + \frac{2(1) + k}{3} = 1$$ ✓ forms equation = 1

$$\frac{2k + 2}{3} = 1$$

$$2k + 2 = 3$$

$$2k = 1$$

$$k = \frac{1}{2}$$ ✓ correct value.

(b) Determine

(i) $P(X = 0)$. (1 mark)

$$P(X=0) = \frac{2(0) + \frac{1}{2}}{3}$$

$$= \frac{1}{6}$$ ✓ correct probability.

(ii) $E(3X - 1)$. (2 marks)

	$(1-p)$	p
x	0	1
$P(X=x)$	$\frac{1}{6}$	$\frac{5}{6}$

Bernoulli

$$E(X) = \frac{5}{6}.$$ ✓ $E(X)$

$$E(3X-1) = 3\left(\frac{5}{6}\right) - 1$$ ✓ $E(3X-1)$

$$= 1.5$$

(iii) $\mathrm{Var}(3X - 1)$. (2 marks)

$$Var(X) = p(1-p)$$

$$= \frac{5}{6}\left(\frac{1}{6}\right)$$ ✓ $Var(X)$

$$= \frac{5}{36}.$$

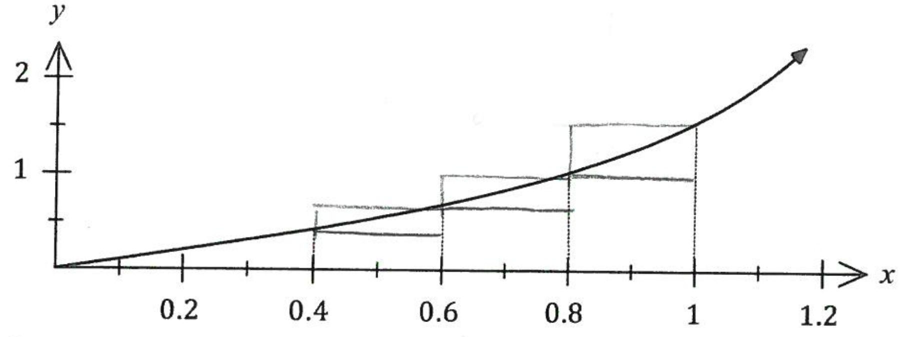
$$Var(3X-1) = 3^2\left(\frac{5}{36}\right)$$ ✓ $Var(3X-1)$

$$= \frac{45}{36}$$

Question 7

(7 marks)

The graph and a table of values for $y = f(x)$ is shown below, where $f(x) = \tan x$.



x	y
0.2	0.2
0.4	0.42
0.6	0.68
0.8	1.03
1	1.56
1.2	2.57

Let $I = \int_{0.4}^{1} \tan x \, dx$.

(a) By using the information shown and considering sums of the form $\sum_i f(x_i)\delta x_i$, explain why $I > 0.426$. (3 marks)

$A = 0.2[f(0.4) + f(0.6) + f(0.8)]$

$= 0.2[0.42 + 0.68 + 1.03]$

$= 0.2(2.13)$

$= 0.426$

✓ shows correct x_i values $\left.\begin{matrix}0.4\\0.6\\0.8\end{matrix}\right\}$

✓ correct sum

A is based on inscribed rectangles which result in an underestimate $\therefore$ $I > 0.426$. ✓ clear explanation

(b) In a similar manner to (a), determine the best estimate for the value of the constant U, where $I < U$. (2 marks)

$A = 0.2[f(0.6) + f(0.8) + f(1)]$

$= 0.2[0.68 + 1.03 + 1.56]$

$= 0.2(3.27)$

$= 0.654$

✓ shows correct x_i values $\left.\begin{matrix}0.6\\0.8\\1\end{matrix}\right\}$

✓ correct sum

(c) Use your previous answers to determine a numerical estimate for I and explain whether your estimate is smaller or larger than the exact value of I. (2 marks)

$\dfrac{0.426 + 0.654}{2} = 0.54$

✓ mean of two estimates

slightly overestimates area as curve is concave upwards.

✓ overestimate with reason stated.

Question 8 **(6 marks)**

The acceleration at time t seconds of a small body travelling in a straight line is given by

$$a(t) = \frac{-12}{\sqrt{4t+5}} \text{ cm/s}^2, \qquad t \geq 0.$$

When $t = 1$ the body was at the origin and 4 seconds later its displacement was 2 cm.

Determine the velocity of the body when $t = 11$.

$t=1 \quad x=0$
change in displacement
$\Delta x = 2$.

$$v(t) = \int \frac{-12}{(4t+5)^{\frac{1}{2}}}\, dt$$
$$= \frac{-12.(4t+5)^{\frac{1}{2}}}{\frac{1}{2}\cdot(4)} + c$$
$$= -6\,(4t+5)^{\frac{1}{2}} + c$$

✓ antiderivative of $a(t)$

$$\Delta x = \int_1^{4+1} v(t)\, dt$$
$$= \int_1^5 -6\,(4t+5)^{\frac{1}{2}} + c \;\; dt.$$

✓ integral for Δx

$$= \left[\frac{-6\,(4t+5)^{\frac{3}{2}}}{\frac{3}{2}\cdot(4)} + ct\right]_1^5$$

✓ antiderivative of $v(t)$

$$= \left[-(4t+5)^{\frac{3}{2}} + ct\right]_1^5$$
$$= -(25)^{\frac{3}{2}} + 5c + 9^{\frac{3}{2}} - c$$
$$= -5^3 + 3^3 + 4c$$
$$\Delta x = -98 + 4c$$

✓ simplifies equation for c

$$\Delta x = 2 \quad \therefore \quad 4c - 98 = 2$$
$$4c = 100$$
$$c = 25$$

✓ uses Δx to determine value of c

$$v(11) = -6\,(4(11)+5)^{\frac{1}{2}} + 25$$
$$= -6\,(49)^{\frac{1}{2}} + 25$$
$$= -42 + 25$$
$$= -17 \text{ cm/s}$$

✓ correct velocity.

UNITS

Supplementary page

Question number: 8 - Alternative solution

$v(t) = -6\sqrt{4t+5} + c$ ✓ correct $v(t)$

$x(t) = \int -6\sqrt{4t+5} + c \, dt$

$= -(4t+5)^{\frac{3}{2}} + ct + d$ ✓ correct $x(t)$

$x(1) = 0 \Rightarrow -(4(1)+5)^{\frac{3}{2}} + c + d = 0$ ✓ equation using $x(1)=0$

$c + d = 27$ ①

$x(5) = 2 \Rightarrow -(4(5)+5)^{\frac{3}{2}} + 5c + d = 2$ ✓ equation using $x(5)=2$

$5c + d = 127$ ②

② - ① $\Rightarrow 4c = 100$

$\underline{c = 25}$ ✓ solves for c

$\therefore v(t) = -6\sqrt{4t+5} + 25$

$v(11) = -6\sqrt{4(11)+5} + 25$

$= -17 \text{ cm s}^{-1}$ ✓ correct $v(11)$

Supplementary page

Question number: ____________________

SHENTON
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Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 & 4

Section Two: Calculator-assumed

Student Name: SOLUTION

Teacher: Ai Friday White

Time allowed for this section

Reading time before commencing work: ten (10) minutes
Working time: one hundred (100) minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templated, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

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Question	**Max.**	**Mark**
9	6	
10	7	
11	7	
12	8	
13	8	
14	6	
15	10	
16	10	
17	8	
18	8	
19	8	
20	6	
Section Two Total	92	
Section Two %		

Section Two: Calculator-assumed (93 Marks)

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.
Working time: 100 minutes.

Question 9 (6 marks)

(a) Function f is defined by $f(x) = 3\log_6(x+6) - 2$ over its natural domain. Determine

(i) the value of the y-intercept of the graph of $y = f(x)$. (1 mark)

$(0, 1)$
$y = 1$

✓ correct value

(ii) the equation of the asymptote of the graph of $y = f(x)$. (1 mark)

$x = -6$

✓ correct EQUATION

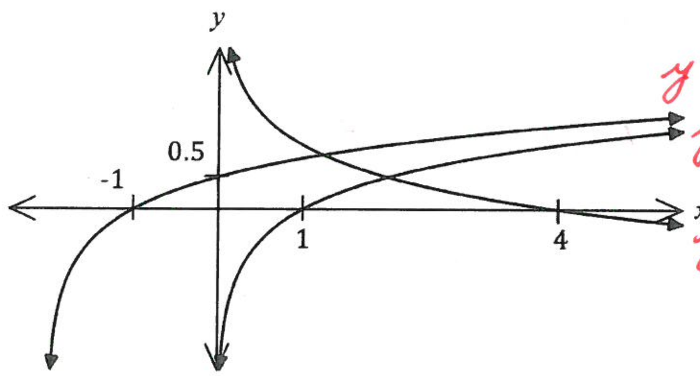
(b) Function g is defined by $g(x) = \log_n x$ over its natural domain, where n is a constant greater than 1.

The graphs shown below have equations $y = g(x)$, $y = a - g(x)$ and $y = g(x+b)$, where a and b are constants.

(i) Label each graph with the appropriate equation from those listed above.

(ii) Determine the value of b and hence determine the value of n and a. (4 marks)

Label graphs with equations $y =$



$y = g(x+b)$
$y = g(x)$
$y = a - g(x)$

✓ all labelled correctly

At $(-1, 0)$ $g(-1) = \log_n(-1+b) = 0$
$n^0 = -1 + b$
$b = 2$

✓ correct value of b

At $(0, 0.5)$ $\log_n(0+2) = 0.5$
$n^{0.5} = 2$
$n = 4$

✓ value of n

At $(4, 0)$ $a - \log_4 4 = 0$
$a = 1$

✓ value of a

Question 10 **(7 marks)**

The percentage distribution of the number of cans of soft drink per order placed with a takeaway food company over a long period of time are shown in the following table.

Number of cans per order	0	1	2	3	4 or more
Percentage of orders	14	24	45	5	12

In the following questions, you may assume that all orders are placed with the company at random and independently.

(a) Determine the probability that the next 10 orders all include at least one can of soft drink. (2 marks)

MUST Give probabilities to 4 or 5 decimal places

$P(X \geq 1) = \frac{86}{100}$

$= 0.86.$

$P = 0.86^{10}$

$= 0.2213$

✓ $P(X \geq 1)$

✓ correct probability at least 4 dp.

(b) During a weekday, a total of 225 orders were placed. Determine the probability that

(i) 40 of these orders included 3 or more cans of soft drink. (3 marks)

$X \sim B(225, 0.17)$

$P(X = 40) = 0.0662$

✓ correct Binomial distribution stated $n = 225$

✓ $p = 0.17$

✓ correct probability

(ii) more than 25 of these orders included no cans of soft drink. (2 marks)

$X \sim B(225, 0.14)$

$P(X \geq 26) = 0.8774$

✓ states Binomial Distribution WITH Parameters

✓ correct probability

Question 11 **(7 marks)**

In a sample of 1 325 university students, 64% said that they never look at their phone while driving.

(a) Show how to use the figures from this sample to construct the 95% confidence interval for the proportion of university students who never look at their phone while driving. (3 marks)

$\hat{p} = 0.64$

$s = \sqrt{\dfrac{(0.64)(0.36)}{1325}}$

$= 0.0132$

✓ standard deviation correct

$C.I = \hat{p} \pm 1.96(s)$

$= 0.64 \pm (1.96)(0.0132)$

$= (0.6142, \; 0.6658)$

✓ show correct use of formula for C.I.

$(0.614, \; 0.666)$

✓ correct C.I to at least 3 dp

(b) According to a newspaper article, "70% of university students never look at their phone while driving". Explain whether the interval from (a) supports this claim. (2 marks)

Interval does not support claim as 0.7 does not lie within the interval.

Careful with interpretations of confidence interval - Not required here anyway!

✓ not supported stated

✓ interval does not include 0.7.

(c) Another source claims that "the majority of university students never look at their phone while driving". Explain whether the interval from (a) supports this claim. (2 marks)

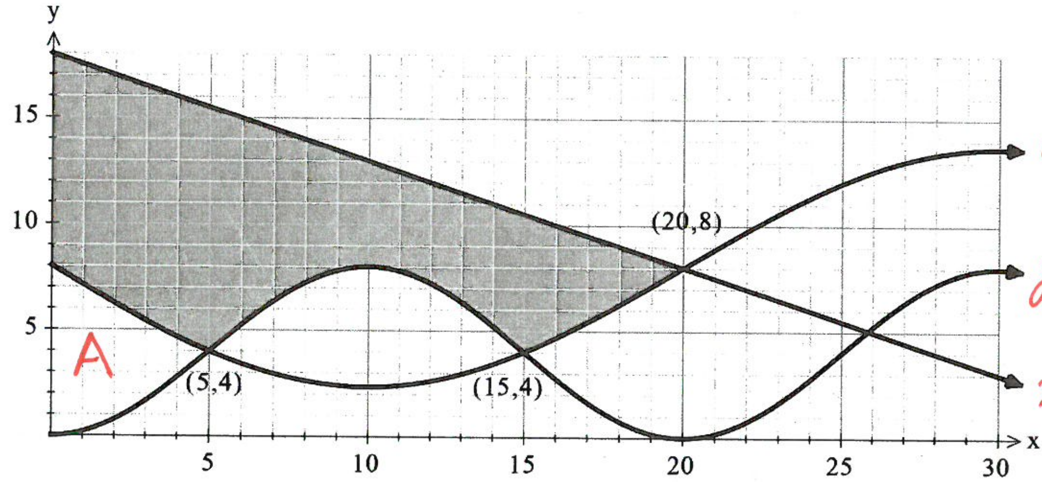
Does support this claim as both lower and upper bound of interval are greater than 0.5

✓ states claim supported

✓ states bounds > 0.5.

Question 12 **(8 marks)**

The diagram shows a flag design, with dimensions in centimetres.



$h(x) = 8 - 4\sqrt{2}\sin\left(\frac{\pi x}{20}\right)$

$g(x) = 4 - 4\cos\left(\frac{\pi x}{10}\right)$

$f(x) = 18 - 0.5x$

The shaded region is bounded by the y-axis, $y = f(x)$, $y = g(x)$ and $y = h(x)$ where

$f(x) = 18 - 0.5x$,

$g(x) = 4 - 4\cos(\frac{\pi x}{10})$ and ✓

$h(x) = 8 - 4\sqrt{2}\sin\left(\frac{\pi x}{20}\right)$.

✓ labels all graphs correctly $f(x)$, $g(x)$, $h(x)$ OK.

(a) Label each graph on the diagram above with the correct function, $f(x), g(x), h(x)$. (1 mark)

(b) Let A be the area of another region on the graph, where $A = \int_0^5 [h(x) - g(x)]\, dx$.

(i) Clearly mark the region on the diagram with the letter A. ✓ A correctly labelled (1 mark)

(ii) Determine the value of A, rounded to one decimal place. (1 mark)

UNIT

$A = 22.2 \text{ cm}^2$ (1dp) ✓ correct A.

(c) Show a calculus method to determine the area of the shaded region. (5 marks)

$A = \underbrace{\int_0^{20} (f(x) - h(x))\, dx}_{A_1} - \underbrace{\int_5^{15} (g(x) - h(x))\, dx}_{A_2}$

✓ $\int_0^{20} f(x) - h(x)\, dx$
✓ $\int_5^{15} g(x) - h(x)\, dx$

$= 172.025 - 36.394$

UNIT

$= 135.63 \text{ cm}^2$

✓ Shows at least one evaluation
✓ $A_1 - A_2$
✓ correct Area

Other valid methods OK

eg $\int_0^5 [f(x) - h(x)]\, dx + \int_5^{15} [f(x) - g(x)]\, dx + \int_{15}^{20} [f(x) - h(x)]\, dx$

Use function names $f(x)$ etc rather than write function in full

Question 13 **(8 marks)**

The heights of girls H in a large study of 3-year-old children are normally distributed with a mean of 94.5 cm and a standard deviation of 3.15 cm.

(a) Determine the probability that a randomly selected girl from the study has a height

(i) that rounds to 93 cm, to the nearest cm. (2 marks)

$P(92.5 < H < 93.5) = 0.1127$

✓ $92.5 < x < 93.5$

✓ correct p to at least 4 dp.

(ii) of at least 90 cm given that they are shorter than 94.5 cm. (2 marks)

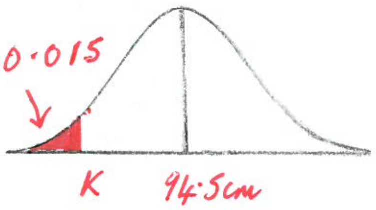
$$P(H \geq 90 / H < 94.5) = \frac{P(90 \leq H < 94.5)}{P(H < 94.5)} = \frac{0.4234}{0.5} = 0.8469$$

✓ indicates correct use of conditional prob

✓ correct prob.

unit

(b) The shortest 1.5% of girls were classified as unusually short. Determine the greatest height of a girl to be classified in this manner. (1 mark)

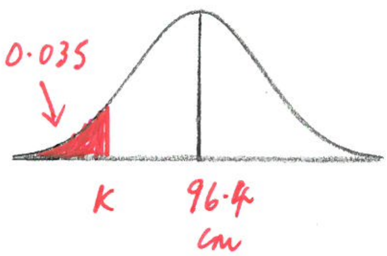


$P(X < K) = 0.015$

$K = 87.66$ cm

✓ correct height.

(c) The heights of boys in the study are normally distributed with mean of 96.4 cm and the shortest 3.5% of boys, with a height less than 90.2 cm, were classified as unusually short. Demonstrate use of the standard normal distribution to determine the standard deviation of the boys' heights. (3 marks)



$P(z < K) = 0.035$

$K = -1.8119.$

✓ Indicates use of z-score to obtain $K = -1.8119$

$$\therefore \frac{90.2 - 96.4}{\sigma} = -1.8119$$

✓ forms equation for σ

$\sigma = 3.42$ cm

✓ correct σ

unit

Demonstrate for 3 marks

If value for σ given and not rounded correctly - wrong - unless full value shown and then rounded incorrectly

Question 14 **(6 marks)**

The voltage *(in volts)* generated by a circuit at time t seconds is given by

$$V(t) = e^{0.2t}\cos(3t) \text{ for } 0 \le t \le 4.$$

(a) Show that the voltage is initially increasing. (2 marks)

$V'(t) = e^{0.2t}\cdot(-3\sin(3t)) + \cos 3t \cdot 0.2e^{0.2t}$ ✓ indicates $V'(t)$

$V'(0) = e^0 \cdot -3\sin(0) + \cos(0)\cdot 0.2e^0$

$= 0.2$ volts/s ✓ shows $V'(0) > 0$

UNIT

$V'(t) > 0 \therefore$ increasing initially

(b) Determine the voltage at the instant the rate of change of voltage first starts to increase. (2 marks)

$V''(t) = 0$

$V''(t) = \dfrac{-224\cos(3t)e^{0.2t} - 30\sin(3t)\cdot e^{0.2t}}{25}$

$V''(t) = 0$

$t = 0.5680$ ✓ ✓ solves $V''(t) = 0$

UNIT $V(0.5680) = -0.1487$ volts ✓ ✓ correct voltage

(c) Show use of the incremental formula to estimate the change in voltage in the one hundredth of a second after $t = 2$. (2 marks)

$\delta V \approx \dfrac{dV}{dt}\cdot \delta t$ $\quad \delta t = 0.01$

$\approx (1.537)(0.01)$ ✓ $\quad \left.\dfrac{dV}{dt}\right|_{t=2} = 1.5370$

≈ 0.0154 Volts ✓

✓ shows use of incremental formula

✓ correct estimate

Question 15 **(10 marks)**

The probability density function for a continuous random variable T is given by:

$$f(t) = \begin{cases} at(t-3) & 0 \le t \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Show use of calculus to determine the value of the constant a. (4 marks)

$\int_0^2 at(t-3)\,dt = 1$

or $\int_0^2 at^2 - 3at \; dt = 1$

✓ correct integral with limits $= 1$

(-1) no "dt"

$a\left[\frac{t^3}{3} - \frac{3t^2}{2}\right]_0^2 = 1$

✓ antiderivative

or $\left[\frac{at^3}{3} - \frac{3at^2}{2}\right]_0^2 = 1$

$\left(\frac{8a}{3} - \frac{12a}{2}\right) - 0 = 1$

✓ substitution

$a = -0.3$ or $\frac{-3}{10}$

✓ $a =$

(b) Determine $P(1 \le T \le 2)$ (2 marks)

$\int_1^2 -0.3t^2 + 0.9t \; dt$

✓ correct definite integral with a from part a).

$= 0.65$ or $\frac{13}{20}$

✓ probability

(c) If $E(T) = \frac{6}{5}$ determine the variance of T. (2 marks)

$\sigma^2 = \int_0^2 \left(t - \frac{6}{5}\right)^2 . f(t)\,dt$

✓ correct integral

$= 0.24$ or $\frac{6}{25}$

✓ σ^2

(d) Find the median of T. (2 marks)

$\int_0^k -0.3t^2 + 0.9t \; dt = 0.5$

✓ correct integral $= 0.5$

$k = 1.238$

✓ answer rounds to 1.24

Question 16 **(10 marks)**

Random samples of 165 people are taken from a large population. It is known that 8% of the population have blue eyes.

(a) Use a discrete probability distribution to determine the probability that the number of people in one sample who have blue eyes is less than 7%. (3 marks)

$X \sim B(165, 0.08)$ ✓ define binomial distr. w correct parameters

7% of 165
$= 11.55$

$P(X < 11.55) = P(X \leq 11)$ or $P(X < 12)$ ✓ correct probability

$\doteq 0.3241$ ✓ answer

(b) Ten consecutive random samples are taken. Determine the probability that the number of those with blue eyes is less than 7% in exactly half of these samples. (2 marks)

$Y \sim B(10, 0.3241)$ ✓ define binomial distribution

$P(Y = 5) = 0.1271$ ✓ correct probability

alternative solution:

${}^{10}C_5 \times (0.3241)^5 \times (1 - 0.3241)^5$ ✓ method clearly shown.

$= 0.1271$ ✓ answer

A large number of random samples of 165 people are taken. The proportion of blue eyed people calculated for each sample and the distribution of these sample proportions analysed.

(c) Describe the continuous probability distribution that these sample proportions approximate, including any parameters. (3 marks)

$\hat{p} = 0.08$ ✓ correct mean

$\sigma = \sqrt{\dfrac{0.08 \times 0.92}{165}}$ ✓ correct σ, σ^2

$= 0.0211$ ✓ normal distribution

$\therefore X \sim N(0.08, 0.0211^2)$

or 0.000446

(d) Describe how two factors affect the closeness of the approximate distribution in (c) to the true distribution of proportions. (2 marks)

- large sample size ✓ indicates large sample size

- p close to 0.5 ✓ indicates p close to 0.5

Question 17 **(8 marks)**

The cross section of a triangular prism with a volume of 54 cm^3 is an equilateral triangle of side length x cm.

(a) Show that the surface area S cm of the prism is given by $S = \frac{\sqrt{3}x^2}{2} + \frac{216\sqrt{3}}{x}$. (4 marks)

(HINT: Use the formula Area of a Triangle = $\frac{1}{2}ab\sin C$.)

$$\text{Area } \Delta = \frac{1}{2} x x \sin 60^\circ = \frac{1}{2} x^2 \frac{\sqrt{3}}{2} = \frac{\sqrt{3}x^2}{4}$$

✓ Area Δ in terms of x

(note: concern that students do not know the properties of a prism)

$$\text{Volume } 54 = \text{Area } \Delta \times h$$

$$h = \frac{54}{\frac{\sqrt{3}x^2}{4}} = \frac{216}{\sqrt{3}x^2} \text{ or } \frac{216\sqrt{3}}{3x^2} \text{ or } \frac{72\sqrt{3}}{x^2}$$

✓ rearrange to express h in terms of x

$$SA = 2 \text{ Area } \Delta + 3xh$$

✓ sum of 2Δ + 3▭

$$= 2\left(\frac{\sqrt{3}x^2}{4}\right) + 3x\left(\frac{72\sqrt{3}}{x^2}\right) = \frac{\sqrt{3}x^2}{2} + \frac{216\sqrt{3}}{x}$$

✓ show substitution incl. clear development of steps

(b) Use calculus to determine the minimum surface area of the triangular prism. (4 marks)

$$\frac{ds}{dx} = \frac{\sqrt{3}x^3 - 216\sqrt{3}}{x^2} \quad \text{or} \quad \sqrt{3}x - \frac{216\sqrt{3}}{x^2}$$

✓ $\frac{ds}{dx}$ (from calc)

$$\frac{ds}{dx} = 0 \text{ when } x = 6$$

✓ equates $\frac{ds}{dx} = 0$ to obtain $x = 6$

$$\left.\frac{d^2s}{dx^2}\right|_{x=6} = \frac{\sqrt{3}x^3 + 432\sqrt{3}}{x^3} = 3\sqrt{3} > 0$$

∴ minimum SA when $x = 6$ cm

✓ justification of minimum

$$SA = 93.53 \text{ cm}^2 \text{ or } 54\sqrt{3} \text{ cm}^2$$

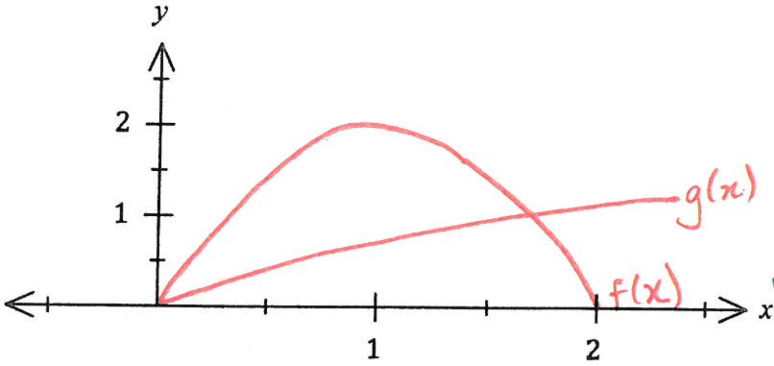
✓ states minimum SA

$\boxed{\text{* UNITS}}$

Question 18 **(8 marks)**

Let $f(x) = 2 - 2(x-1)^2$ and $g(x) = \ln(x+1)$.

(a) Sketch the graphs of $y = f(x)$ and $y = g(x)$ for $x \geq 0$ on the axes below. (2 marks)



✓ graphs $y = g(x)$

✓ graphs $y = f(x)$

(b) Show that $\frac{d}{dx}\big((x+1)\ln(x+1) - (x+1)\big) = \ln(x+1)$. (2 marks)

$= (x+1)\left(\frac{1}{x+1}\right) + \ln(x+1)(1) - 1$

$= \frac{x+1}{x+1} + \ln(x+1) - 1$

$= 1 + \ln(x+1) - 1$

$= \ln(x+1)$

✓ uses product rule correctly

✓ differentiates $\ln(x+1)$ term correctly

(c) Show that the area of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$, and the straight line $x = 1$ is exactly $\frac{7}{3} - 2\ln 2$ square units. (4 marks)
(HINT: use your answer in part (b).)

$A = \int_0^1 f(x) - g(x)\, dx$

$= \int_0^1 2 - 2(x-1)^2 - \ln(x+1)\, dx$

$= \left[2x - \frac{2(x-1)^3}{3} - \big((x+1)\ln(x+1) - (x+1)\big)\right]_0^1$

or

$\left[-\frac{2x^3}{3} + 2x^2 - \big((x+1)\ln(x+1) - (x+1)\big)\right]_0^1$

$= (2 - 0 - 2\ln 2 + 2) - (0 + \frac{2}{3} - \ln 1 + 1)$

$= 2 - 2\ln 2 + 2 - \frac{2}{3} - 1$

$= \frac{7}{3} - 2\ln 2$

✓ writes correct integral for area

✓ antidifferentiates f(x) correctly

✓ g(x) correctly

✓ show correct substitution and simplification

*if used calc to antidifferentiate area, max 3 marks.

Question 19 **(8 marks)**

A customer plays an online game of chance. In this game, the computer randomly picks one letter from the letters of the word **LUCKY**, one letter from the letters of the word **BOIST**, and one letter from the letters of the word **GAMER**.
For example, the computer might pick **K** from LUCKY, **S** from BOIST and **R** from GAMER, making **KSR.**
The customer can see the words but does not know the computer's 3-letter picks and has to guess the letter it has chosen from each word. The customer can guess 0 letters, 1 letter, 2 letters or 3 letters correctly.

The random variable X is the number of letters correctly guessed by a customer in one play of the game.

(a) Complete the table below to show the probability distribution of X. (3 marks)

x	0	1	2	3
$P(X=x)$	0.512	0.384	0.096	0.008

$X \sim B(3, \frac{1}{5})$ Use Binomial

✓ use of Binomial with parameters

OR

$P(X=3) = (\frac{1}{5})(\frac{1}{5})(\frac{1}{5}) = \frac{1}{125}$

$P(X=2) = (\frac{1}{5})(\frac{1}{5})(\frac{4}{5}) \times {}^3C_2 = \frac{12}{125}$

$P(X=1) = (\frac{1}{5})(\frac{4}{5})(\frac{4}{5}) \times {}^3C_1 = \frac{48}{125}$

✓ one correct

$P(X=0) = (\frac{4}{5})^3 = \frac{64}{125}$

✓ all correct

Each game costs a player 25 cents. A player wins a prize of \$14 if they guess all three letters correctly, \$1.40 if they guess two out of three letters correctly but otherwise wins nothing.

(b) Determine $E(Y)$ and $\mathrm{Var}(Y)$, where the random variable Y is the gain, in cents, made by the customer in one play of the game. (4 marks)

Gain \$	14−0.25 = \$13.75	1.40−0.25 \$1.15	0−0.25 −\$0.25
Y Gain cents	1375	115	−25
$P(Y=y)$	0.008	0.096	0.896

✓ correct values for Y

✓ indicates distribution of Y

$E(Y) = 1375(0.008) + 115(0.096) + -25(0.896)$

$= -0.36$ cents.

✓ correct mean

$\mathrm{Var}(Y) = (130.21)^2$ from calculator

$= 16954$ cents.^{2}

✓ correct variance

(c) If 1 000 people played the game, calculate the expected gross profit for the game owners. (1 mark)

Profit for game owners long term

$1000 \times 0.36c = 360c$

$= \$3.60$

✓ correct revenue in \$'s.

Question 20 **(6 marks)**

A student was set the task of determining the proportion of people in their suburb who use public transport at least once a week.

(a) Briefly discuss the main source of bias in each of the following sampling methods.

(i) The student invites people via social media to respond to their survey. (1 mark)

eg. Volunteer sampling / Self Selection sampling ∴ some of population have zero chance of selection.

eg. People who respond may not live in suburb ∴ not representative of suburb population. ✓ indicates one source/type of BIAS and WHY not.

(ii) The student asks everyone she meets until she has a large enough sample. (1 mark)

eg. Convenience sampling - no regard for need of sample to represent population

eg. May only be close to where student lives, not represent whole suburb. ✓ indicates one source/type of BIAS. and WHY not.

(b) The student noted that 39 out of all those sampled said they used public transport at least once a week and went on to construct the confidence interval $(0.49, 0.81)$. Determine the level of confidence of this interval. (4 marks)

$$\hat{p} = \frac{0.49+0.81}{2} = 0.65$$

$$M.E = 0.81 - 0.65 = 0.16$$

✓ calculates $\hat{p}$ and M.E.

$$\frac{39}{n} = 0.65$$

$$n = 60$$

✓ calculates sample size, n

$$\therefore \quad 0.16 = z\sqrt{\frac{0.65(0.35)}{60}}$$

$$z = 2.598$$

✓ shows calculation for z score

$$P(-2.598 < Z < 2.598) = 0.9906$$

Level of confidence 99.06%.

✓ level of confidence

End of Questions

Supplementary page

Question number:____________